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Countable connected Hausdorff and Urysohn bunches of arcs in the plane

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Abstract

In this paper, we answer a question by Krasinkiewicz, Reńska and Sobolewski by constructing countable connected Hausdorff and Urysohn spaces as quotient spaces of bunches of arcs in the plane. We also consider a generalization of graphs by allowing vertices to be continua and replacing edges by not necessarily connected sets. We require only that two "vertices" be in the same quasi-component of the "edge" that contains them. We observe that if a graph G cannot be embedded in the plane, then any generalized graph modeled on G is not embeddable in the plane. As a corollary we obtain not planar bunches of arcs with their natural quotients Hausdorff or Urysohn. This answers another question by Krasinkiewicz, Reńska and Sobolewski.

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1. Introduction

The following notion of countable bunches of arcs was extensively studied in a recent paper by Krasinkiewicz, Reńska and Sobolewski [5]. A *countable bunch of arcs* is a metric space X admitting a partition into a countable collection $\mathcal{A}$ of mutually exclusive arcs. Since any countable partition of a continuum is trivial (a theorem of Sierpiński, see

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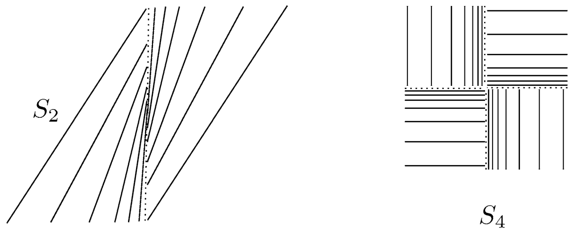

Fig. 1.

[6, p. 173]), the collection $\mathcal{A}$ is unique and depends only on X. Elements of $\mathcal{A}$ are maximal arcs contained in X. We say that a set $B \subset X$ is *saturated* by $\mathcal{A}$ if $A \subset B$ for each $A \in \mathcal{A}$ such that $A \cap B \neq \emptyset$. By the *natural quotient* of X, denoted $X/\sim$, we understand $\mathcal{A}$ with the quotient topology. Let q denote the natural projection of X onto $X/\sim = \mathcal{A}$. The projection q shrinks each arc in X to a point. Subsets of X saturated by $\mathcal{A}$ coincide with preimages by q of sets in $X/\sim$. A set $U \subset X/\sim$ is open if and only if $q^{-1}(U)$ is open in X. Therefore, each open set U in $X/\sim$ corresponds to $q^{-1}(U)$ which is open in X and saturated by $\mathcal{A}$. If a topological space Y is homeomorphic to $X/\sim$ for a certain countable bunch of arcs X, we will call X a *resolution* of Y.

For example, consider the bunches of arcs $S_1, S_2, \ldots$ described in [5] and called there Sierpiński's bunches (S_1 is described in [9], cf. [6, p. 175]). The bunch S_n consists of n infinite groups of arcs. The groups are numbered from 0 to $n-1$. The closure of the ith group contains one endpoint of each arc in the jth group where $j = i + 1 \bmod n$ (see Fig. 1). Observe that each non-empty open subset of S_n saturated by $\mathcal{A}$ must contain all but finitely many of the arcs. Therefore, $S_n/\sim$ is a countable infinite set with the cofinite topology (a set is open in $S_n/\sim$ if and only if either it is the entire space or its complement is finite). The bunches $S_1, S_2, \ldots$ are, therefore, different resolutions of the same space (see [5]).

We will say that an arc A is *free* in X if A without its endpoints is open in X. We will say that X is a *free bunch of arcs* if each arc in $\mathcal{A}$ is free in X. Observe that each of the Sierpiński bunches S_n is contained in the plane and it is free. Observe also that for each $n > 1$, the bunch S_n may be constructed in such a way that each arc in $\mathcal{A}$ is straight linear segment.

Another interesting example of a connected countable bunch of straight linear segments in the plane was constructed by Knaster, Lelek and Mycielski in [4, Example 3]. The bunch defined there is locally connected, and even though it is not free, its construction may be modified to get a locally connected free bunch of straight linear segments.

We will say that a countable bunch of arcs X is *Hausdorff* (*Urysohn*) if $X/\sim$ is a Hausdorff (*Urysohn*) space. (A topological space is a Urysohn space provided that each two different points have neighborhoods with disjoint closures.) Clearly, S_n is not Hausdorff. It may be also proven that the bunch defined in [4, Example 3] is not Hausdorff either. Since the quotient map is continuous, $X/\sim$ is connected for a connected bunch of arcs X. It was observed in [5] that $X/\sim$ cannot be regular if X is connected (see also [2, 1.5.17]).

In [5], Krasinkiewicz, Reńska and Sobolewski characterized T_1-spaces that have resolutions into bunches of arcs. It follows from the characterization that the classic examples of countable connected Hausdorff/Urysohn spaces by Bing [1], Roy [8], and Jones and

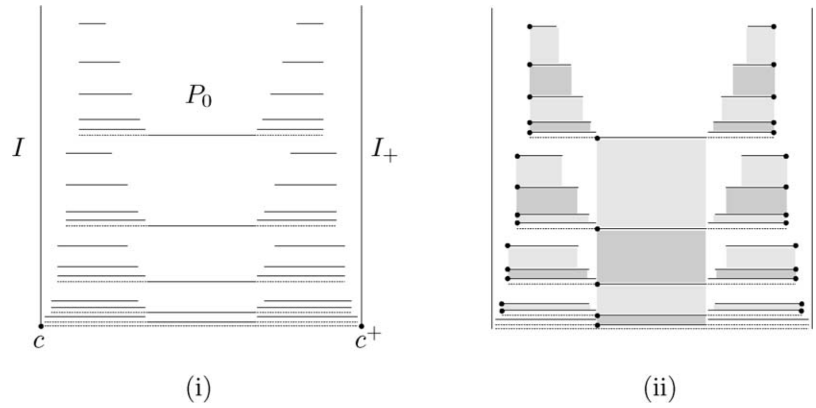

Fig. 2.

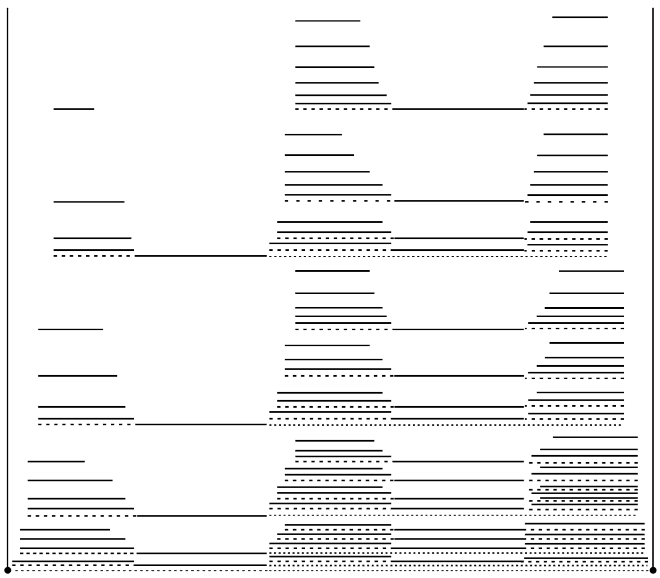

Fig. 3.

Stone [3] are quotients of countable bunches of arcs in $\mathbb{R}^3$. Krasinkiewicz, Reńska and Sobolewski asked if there exists an infinite countable connected bunch of arcs in the plane with the natural quotient Hausdorff (or Urysohn) [5, Problem 1]. We answer this question in both versions by constructing connected countable bunches of free straight linear segments, one Hausdorff and not Urysohn (see 3.21), and one Urysohn (see 3.22). We obtain two different examples from the same construction by using two not connected bunches (shown in Figs. 2(i) and 3) as basic building blocks. In a manner similar to the construction of [4, Example 3], we achieve connectedness by pasting together infinitely many copies of the starting not connected bunch. Fig. 4 shows the bunch resulting from the construc-

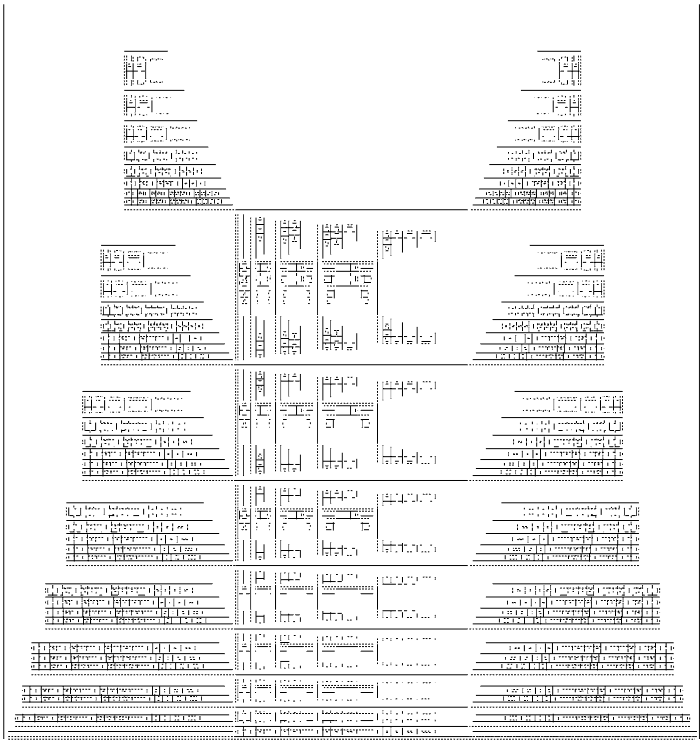

Fig. 4.

tion with the basic building block from Fig. 2(i). The bunch in Fig. 4 is Hausdorff but not Urysohn. We get an Urysohn bunch by starting from the basic building block depicted in Fig. 3.

Even though the idea of our construction is simple (see the beginning of Section 2), the precise description of our examples is much more complicated then that of the classic connected countable Hausdorff and Urysohn spaces by Bing, Roy, and Jones and Stone. In return, we have “nice” (see Fig. 4) geometric planar precursors for “strange” topological spaces.

In Section 4, we introduce a notion of a *quasi-graph* (see Definition 4.1). Roughly speaking, a quasi-graph is a generalization of a graph with continua instead of vertices and closed not necessarily connected sets instead of edges. We require that two “vertices” be in the same quasi-component of the “edge” that contains them. (Recall that the *quasi-component* of a point x in a space X is the intersection of all closed and open sets containing x.) We observe that if a graph G cannot be embedded in the plane, then any quasi-graph modeled on G cannot be embedded in the plane (Theorem 4.6). As a corollary we obtain not planar bunches of arcs with their natural quotients Hausdorff or Urysohn (see 4.7). This answers another question by Krasinkiewicz, Reńska and Sobolewski [5, Problem 3(b)].

2. Basic building blocks for Hausdorff and Urysohn bunches in the plane

In the next section we construct connected Hausdorff and Urysohn bunches of free straight linear segments contained in the plane $\mathbb{R}^2$ (see Fig. 4). The idea of the construction is very simple, but the precise notation of the examples is somewhat cumbersome, since the bunches are fractal-like and are assembled inductively by inserting infinitely many elements of a basic building block in each step of the construction. We advise the reader to look at the following sketch of the construction and understand its simple idea with help of the figures. We use the same idea to obtain two different examples of connected bunches of free straight linear segments in $\mathbb{R}^2$. The first of them is Hausdorff (but not Urysohn), and the second is Urysohn. We achieve this variation by using two different basic building blocks $\widetilde{P}_0$ (see Fig. 2(i)) and $\widetilde{R}_0$ (see Fig. 3). We list the required properties in Definition 2.4, and then use a generic basic building block $\widetilde{X}_0$ satisfying 2.4 in our construction.

Let N denote the set on non-negative integers. For any two points a and b in the plane $\mathbb{R}^2$, let $\langle a, b\rangle$ denote the straight linear segment joining a and b. Let S denote the unit square $[0,1]\times[0,1]$, and let $c=(0,0)$, $c^+=(1,0)$, $I=\{0\}\times[0,1]$ and $I_+=\{1\}\times[0,1]$.

Sketch of the construction. To make this sketch more readable, we illustrate the construction using the simpler of our two basic blocks. The other basic block is just more complicated, but the idea of the construction is essentially the same.

We begin the construction with two parallel straight linear segments I and I_+. Between those two segments we insert a countable bunch of arcs P_0 as shown in Fig. 2(i). The bunch P_0 (see Example 2.1 for precise description) consists of straight linear horizontal segments. The bunch P_0 is so constructed that its closure intersects $I\cup I_+$ at two points c and c^+, where c is the lower endpoint of I and c^+ is the lower endpoint of I_+. Denote the union $I\cup P_0\cup I_+$ by $\widetilde{P}_0$. The main idea of the construction is to have I and I_+ contained in the same quasi-component of $\widetilde{P}_0$ (see Proposition 2.2). On the other hand, $\widetilde{P}_0$ is a Hausdorff bunch of arcs. For instance, in order to find disjoint open neighborhoods of I and I_+, respectively, that are saturated by the segments in $\widetilde{P}_0$, take the segments in $\widetilde{P}_0$ contained (respectively) in the left and the right one third vertical strips in the rectangle between I and I_+. We use P_0 as a basic building block. We inductively paste together infinitely many mutually exclusive copies of P_0 to get a connected Hausdorff bunch of free straight linear segments.

Fig. 2(ii) depicts $\widetilde{P}_0$ with a countable collection of gray rectangles whose interiors are mutually exclusive and do not intersect $\widetilde{P}_0$. The union of the rectangles and $\widetilde{P}_0$ is connected. We insert a copy of $\widetilde{P}_0$ into each of the rectangles such that c and c^+ are taken to the corners indicated with the black dots. This makes $\widetilde{P}_0$ to be in one quasi-component of so constructed bunch of segments. Then we insert a copy of $\widetilde{P}_0$ into each of the gray rectangles associated with previously inserted copies of $\widetilde{P}_0$. The new insertions cause that the bunch of segments constructed previously is contained in one quasi-component of the bunch in the current level of construction. We continue this process infinitely guaranteeing connectedness of the resulting bunch of arcs $\widetilde{P}$ (see Fig. 4). The bunch $\widetilde{P}$ is Hausdorff but not Urysohn, since $\widetilde{P}_0$ is not Urysohn. We obtain an Urysohn bunch by replacing $\widetilde{P}_0$ by an Urysohn basic bunch depicted in Fig. 3.

Example 2.1 *(Construction of $\widetilde{P}_0$).* Let $u_0 < 0.1, u_1, u_2, \ldots$ be a strictly decreasing sequence of real numbers converging to 0. Let $t_0 < 1, t_1, t_2, \ldots$ be another strictly decreasing sequence also converging to 0. Set $t_{-1} = 1$. For each $j \in N$, let $t_{j,0} < t_{j-1}, t_{j,1}, t_{j,2}, \ldots$ be a strictly decreasing sequence converging to t_j. Set $t_{j,-1} = t_{j-1}$.

We begin the construction with two parallel straight linear segments I and I_+, the vertical sides of the unit square S. We insert between I and I_+ a countable bunch of arcs P_0 as shown in Fig. 2(i). The bunch P_0 consists of straight linear horizontal segments located in three vertical strips. The middle one-third vertical strip contains a sequence of segments $I_0^{(1)}, I_1^{(1)}, I_2^{(1)}, \ldots$ where $I_j^{(1)} = \langle a_j^{(1)}, b_j^{(1)} \rangle$ with $a_j^{(1)} = (1/3, t_j)$ and $b_j^{(1)} = (2/3, t_j)$. For each such centrally located segment $I_j^{(1)}$, the bunch P_0 contains two additional sequences of segments. The segments $I_{j,0}^{(0)}, I_{j,1}^{(0)}, I_{j,2}^{(0)}, \ldots$ are located in the left one third vertical strip between I and I_+, and their right endpoints converge to the left endpoint of the segment $I_j^{(1)}$. The segments $I_{j,0}^{(2)}, I_{j,1}^{(2)}, I_{j,2}^{(2)}, \ldots$ are located in the right one third vertical strip between I and I_+, and their left endpoints converge to the right endpoint of $I_j^{(1)}$. More precisely, $I_{j,k}^{(0)} = \langle a_{j,k}^{(0)}, b_{j,k}^{(0)} \rangle$ where $a_{j,k}^{(0)} = (u_j, t_{j,k})$ and $b_{j,k}^{(0)} = (1/3 - u_k, t_{j,k})$. Similarly, $I_{j,k}^{(2)} = \langle a_{j,k}^{(2)}, b_{j,k}^{(2)} \rangle$ where $a_{j,k}^{(2)} = (2/3 + u_k, t_{j,k})$ and $b_{j,k}^{(2)} = (1 - u_j, t_{j,k})$. Let $\mathcal{I}_0$ be the collection of all of the arcs $I_{j,k}^{(0)}$, $I_j^{(1)}$ and $I_{j,k}^{(2)}$ where $j, k \in N$. We define

$$P_0 = \bigcup_{A \in \mathcal{I}_0} A$$

and $\widetilde{P}_0 = P_0 \cup I \cup I_+$. Also, set $\widetilde{\mathcal{I}}_0 = \mathcal{I}_0 \cup \{I, I_+\}$.

Proposition 2.2. *In Example* 2.1*, each element of $\mathcal{I}_0$ is a quasi-component of $\widetilde{P}_0$. I and I_+ are contained in the same quasi-component of $\widetilde{P}_0$.*

Proof. To prove the first part of the proposition, observe that $I_{j,k}^{(0)}$ and $I_{j,k}^{(2)}$ are closed and open in $\widetilde{P}_0$. Also, the set $G_m = I_j^{(1)} \cup \bigcup_{k \geqslant m} (I_{j,k}^{(0)} \cup I_{j,k}^{(2)})$ is closed and open in $\widetilde{P}_0$ for each $m \in N$, and $\bigcap_{m \in N} G_m = I_j^{(1)}$.

To prove the second part of the proposition, take U a closed and open set in $\widetilde{P}_0$ intersecting I. Since I is connected, it is contained in U. We will prove that I_+ is also contained in U. Suppose, to the contrary, that I_+ intersects $U_+ = \widetilde{P}_0 \setminus U$. Then, $I_+ \subset U_+$. Since $c \in U$, there is an integer ℓ such that $a_{j,k}^{(0)} \in U$ for each $j \geqslant \ell$ and each $k \in N$. Similarly, there is an integer ℓ_+ such that $b_{j,k}^{(2)} \in U_+$ for each $j \geqslant \ell_+$ and each $k \in N$. Let m be the maximum of ℓ and ℓ_+. Since $a_{m,k}^{(0)} \in U$ for each $k \in N$, $I_{m,k}^{(0)} \subset U$ and, consequently, $b_{m,k}^{(0)} \in U$ for each $k \in N$. Thus, $a_m^{(1)} \in U$ because U is closed. Similarly, $a_{m,k}^{(2)} \in U_+$ for each $k \in N$, and $b_m^{(1)} \in U_+$. Now, $I_m^{(1)}$ intersects both U and U_+ which is impossible. □

Remark 2.3. Suppose that U and U_+ are open subsets of $\widetilde{P}_0$, saturated by $\widetilde{\mathcal{I}}_0$, and containing I and I_+, respectively. By following the same argument as in the preceding proof, we infer that the closures of U and U_+ must both intersect $I_m^{(1)}$ for some $m \in N$. It follows that $\widetilde{P}_0$ is not an Urysohn bunch.

Definition 2.4. We will now state 13 conditions that must be satisfied by a basic building block used in our construction. If the following conditions (1)–(13) are satisfied, the bunch of free straight linear segments resulting from the construction will be connected and Hausdorff. The bunch will be Urysohn if the additional condition (14) holds. [To make this lengthy definition easier to read, we ask the reader to simultaneously check that the conditions (1)–(13) are satisfied by the bunch $\widetilde{P}_0$ defined in Example 2.1. We include some comments to help with this check. We enclose the additional comments in square brackets to distinguish them from the text of the definition.]

Suppose that

(1) $\mathcal{I}_0 = \{I_0, I_1, \ldots\}$ is a collection of mutually disjoint straight linear horizontal segments contained in the interior of S,
(2) $\mathcal{S}_0 = \{S_0, S_1, \ldots\}$ is a collection of rectangles contained in the interior of S, and having their sides horizontal and vertical,
(3) the sequence of heights of S_n converges to 0,
(4) I_n is the upper side of S_n for each $n \in N$,
(5) $g : N \to N$ is an injection such that, for each $n \in N$, the lower side of S_n is contained in $I_{g(n)}$, and
(6) if $I_k \cap S_n \neq \emptyset$ for some $k, n \in N$, then either $k = n$ or $k = g(n)$.

[In case of Example 2.1, let $I_0, I_1, \ldots$ be any enumeration of $\mathcal{I}_0$. For each $n \in N$ let $g(n)$ be such integer that $I_{g(n)}$ is the first segment in $\mathcal{I}_0$ you encounter going downward from I_n. Let S_n be the rectangle with its top side I_n and the base contained in $I_{g(n)}$. The rectangles are illustrated in Fig. 2(ii).]

Set $X_0 = \bigcup_{n \in N} I_n$, $\widetilde{\mathcal{I}}_0 = \mathcal{I}_0 \cup \{I, I_+\}$ and $\widetilde{X}_0 = I \cup X_0 \cup I_+$. We will require that

(7) $I \cap \mathrm{Cl}(X_0) = \{c\}$ and $I_+ \cap \mathrm{Cl}(X_0) = \{c^+\}$.
(8) Each element of $\mathcal{I}_0$ is a quasi-component of $\widetilde{X}_0$. I and I_+ are contained in the same quasi-component of $\widetilde{X}_0$. [See 2.2.]
(9) For each non-empty set $M \subset N$, if $g(M) \cup g^{-1}(M) \subset M$ and the set $\bigcup_{m \in M} I_m$ is closed and open in X_0, then $M = N$.

[In Example 2.1, the union $\bigcup_{n \in N} S_n$ is connected (see Fig. 2(ii)). Observe that if M is as in (9), then $\bigcup_{m \in M} S_n$ must be closed and open in $\bigcup_{n \in N} S_n$. Hence, (9) holds for Example 2.1.]

Let a_n and b_n denote the endpoints of I_n. In each of the rectangles S_n choose two corner points c_n and c_n^+ belonging to one of the two vertical sides of S_n so that c_n is above c_n^+. [In Example 2.1, let a_n be always the left endpoint of I_n. Set $c_n = a_n$ and $c_n^+ = a_{g(n)}$ if I_n is one of the segments $I_{j,k}^{(0)}$ or $I_j^{(1)}$. Set $c_n = b_n$ and $c_n^+ = b_{g(n)}$ if I_n is one of the segments $I_{j,k}^{(2)}$. c_n and c_n^+ are represented by the black dots in Fig. 2(ii).]

In the following conditions (10)–(12), n is an arbitrary element of N.

(10) c_n is one of the endpoints of I_n.
(11) c_n^+ is one of the endpoints of $I_{g(n)}$.

Let $\hat{g}(n) = g^{-1}(n)$ if $n \in g(N)$ and let $\hat{g}(n) = n$ otherwise. We will require that

(12) $I_n \setminus \{a_n, b_n\}$ does not intersect the closure of $\bigcup_{k \in N \setminus \{n, \hat{g}(n)\}} S_k$.

We will say that a set $W \subset \widetilde{X}_0$ is *bisaturated* by $\widetilde{\mathcal{I}}_0$ if W is saturated by $\widetilde{\mathcal{I}}_0$ and it contains both $I_{g(n)}$ and $I_{\hat{g}(n)}$ for each $n \in N$ such that $I_n \subset W$.

(13) There are two sets G and G_+ open in $\widetilde{X}_0$ such that
  (a) G and G_+ are bisaturated by $\widetilde{\mathcal{I}}_0$,
  (b) $I \subset G$, $I_+ \subset G_+$, and
  (c) $G \cap G_+ = \emptyset$.

[For $\widetilde{X}_0 = \widetilde{P}_0$ in Example 2.1, define $G = I \cup \bigcup_{j,k \in N} I_{j,k}^{(0)}$ and $G_+ = I_+ \cup \bigcup_{j,k \in N} I_{j,k}^{(2)}$.]

We will say that $\widetilde{X}_0$ is a *basic Hausdorff bunch* if the conditions (1)–(13) are satisfied. We will say that $\widetilde{X}_0$ is a *basic Urysohn bunch* if it is a basic Hausdorff bunch satisfying the following additional condition.

(14) There are two sets F and F_+ closed in $\widetilde{X}_0$ such that
  (a) F and F_+ are saturated by $\widetilde{\mathcal{I}}_0$,
  (b) $G \subset F$, $G_+ \subset F_+$, and
  (c) $F \cap F_+ = \emptyset$.

Observe that (8) and (13) imply that each basic Hausdorff bunch is a Hausdorff bunch. By (8) and (14) each basic Urysohn bunch is a Urysohn bunch.

It will be convenient to consider the two following additional conditions (15) and (16). Observe that (15) follows from (4) and (5), while (16) is a consequence of (6).

(15) $S_n \cap S_{g(n)}$ is the lower side of S_n for each $n \in N$.
(16) If $S_k \cap S_n \neq \emptyset$ for some $k, n \in N$, then either $k = n$ or $k = g(n)$ or $n = g(k)$.

Corollary 2.5. $\widetilde{P}_0$ *from Example* 2.1 *is a basic Hausdorff bunch.*

Example 2.6 *(Construction of $\widetilde{R}_0$).* We construct here a basic Urysohn bunch $\widetilde{R}_0$ (see Fig. 3).

Let u_j, t_j and $t_{j,k}$ be as defined in Example 2.1. For each $j, k \in N$, let $t_{j,k,0} < t_{j,k-1}, t_{j,k,1}, t_{j,k,2}, \ldots$ be a strictly decreasing sequence converging to $t_{j,k}$.

Recall that in Example 2.1 the square S was divided into three vertical strips, each containing different types of segments comprising P_0. This time consider five vertical strips $S^{(0)} = (0, 1/5) \times [0, 1]$, $S^{(1)} = [1/5, 2/5] \times [0, 1]$, $S^{(2)} = (2/5, 3/5) \times [0, 1]$, $S^{(3)} = [3/5, 4/5] \times [0, 1]$ and $S^{(4)} = (4/5, 1) \times [0, 1]$. The bunch R_0 consists of straight linear horizontal segments, each contained in one of the strips. As it was in Example 2.1, we denote the segments by $I_*^{(*)}$ where the superscript is the number of the strip $S^{(*)}$ containing $I_*^{(*)}$. (Note that $I_*^{(*)}$ will be redefined here and will denote different segments than in 2.1.)

The strip $S^{(1)}$ contains a sequence of segments $I_0^{(1)}, I_1^{(1)}, I_2^{(1)}, \ldots$ where $I_j^{(1)} = \langle a_j^{(1)}, b_j^{(1)} \rangle$ with $a_j^{(1)} = (1/5, t_j)$ and $b_j^{(1)} = (2/5, t_j)$.

The segments contained in $S^{(0)}$ and $S^{(3)}$ are subscripted by two integers $j, k \in N$. For each $j, k \in N$, $S^{(0)}$ contains the segment $I_{j,k}^{(0)} = \langle a_{j,k}^{(0)}, b_{j,k}^{(0)} \rangle$ where $a_{j,k}^{(0)} = (u_j, t_{j,k})$ and $b_{j,k}^{(0)} = (1/5 - u_k, t_{j,k})$. Similarly, the segment $I_{j,k}^{(3)} = \langle a_{j,k}^{(3)}, b_{j,k}^{(3)} \rangle$, where $a_{j,k}^{(3)} = (3/5, t_{j,k})$ and $b_{j,k}^{(3)} = (4/5, t_{j,k})$, is contained in $S^{(3)}$ for each $j, k \in N$.

The segments contained in $S^{(2)}$ and $S^{(4)}$ are subscripted by three integers $j, k, i \in N$. For each $j, k, i \in N$, set $I_{j,k,i}^{(2)} = \langle a_{j,k,i}^{(2)}, b_{j,k,i}^{(2)} \rangle$, where $a_{j,k,i}^{(2)} = (2/5 + u_k, t_{j,k,i})$ and $b_{j,k,i}^{(2)} = (3/5 - u_i, t_{j,k,i})$. Also, set $I_{j,k,i}^{(4)} = \langle a_{j,k,i}^{(4)}, b_{j,k,i}^{(4)} \rangle$, where $a_{j,k,i}^{(4)} = (4/5 + u_i, t_{j,k,i})$ and $b_{j,k,i}^{(4)} = (1 - u_j, t_{j,k,i})$.

Let $\mathcal{I}_0$ be the collection of all of the arcs $I_{j,k}^{(0)}$, $I_j^{(1)}$, $I_{j,k,i}^{(2)}$, $I_{j,k}^{(3)}$ and $I_{j,k,i}^{(4)}$ where $j, k, i \in N$. We define

$$R_0 = \bigcup_{A \in \mathcal{I}_0} A$$

and $\widetilde{R}_0 = R_0 \cup I \cup I_+$. Also, set $\widetilde{\mathcal{I}}_0 = \mathcal{I}_0 \cup \{I, I_+\}$.

Proposition 2.7. *In Example* 2.6*, each element of* $\mathcal{I}_0$ *is a quasi-component of* $\widetilde{R}_0$*.* I *and* I_+ *are contained in the same quasi-component of* $\widetilde{R}_0$*.*

Proof. To prove the first part of the proposition, take an arbitrary $A \in \mathcal{I}_0$. Observe that A is closed and open in $\widetilde{R}_0$ if is contained in $S^{(0)} \cup S^{(2)} \cup S^{(4)}$ (see Fig. 3).

If $A = I_j^{(1)}$ is contained in $S^{(1)}$, then the set $G_j^m = R_0 \cap [0, 1] \times [t_j, t_{j,m})$ is saturated by $\mathcal{I}_0$ and closed and open in $\widetilde{R}_0$ for each $m \in N$. Observe that $\bigcap_{m \in N} G_j^m = I_j^{(1)}$.

Finally, if $A = I_{j,k}^{(3)}$ is contained in $S^{(3)}$, then the set $G_{j,k}^m = R_0 \cap (2/5, 1) \times [t_{j,k}, t_{j,k,m})$ is saturated by $\mathcal{I}_0$ and closed and open in $\widetilde{R}_0$ for each $m \in N$. Observe that $\bigcap_{m \in N} G_{j,k}^m = I_{j,k}^{(3)}$.

To prove the second part of the proposition, take U a closed and open set in $\widetilde{R}_0$ intersecting I. Set $U_+ = \widetilde{R}_0 \setminus U$ and observe that both U and U_+ are saturated by $\widetilde{\mathcal{I}}_0$. We need to prove that I_+ is contained in U. Suppose, to the contrary, that $I_+ \subset U_+$.

Since $c \in U$, there is an integer ℓ such that $a_{j,k}^{(0)} \in U$ for each $j \geqslant \ell$ and each $k \in N$. Consequently, $I_{j,k}^{(0)} \subset U$ for each $j \geqslant \ell$ and each $k \in N$. Since $\lim_{k \to \infty} b_{j,k}^{(0)} = a_j^{(1)}$ and U is closed, $I_j^{(1)} \subset U$ for each $j \geqslant \ell$.

Since $c^+ \in U_+$, there is an integer ℓ_+ such that $b_{j,k,i}^{(4)} \in U_+$ for each $j \geqslant \ell_+$ and each $k, i \in N$. Consequently, $I_{j,k,i}^{(4)} \subset U_+$ for each $j \geqslant \ell_+$ and each $k, i \in N$. Since $\lim_{i \to \infty} a_{j,k,i}^{(4)} = b_{j,k}^{(3)}$ and U_+ is closed, $I_{j,k}^{(3)} \subset U_+$ for each $j \geqslant \ell_+$ and each $k \in N$.

Let m be the maximum of ℓ and ℓ_+. Since $b^{(1)}_m \in U$ and U is open, there is $p \in N$ such that $a^{(2)}_{m,k,i} \in U$ for each $k \geqslant p$ and each $i \in N$. Consequently, $I^{(2)}_{m,p,i} \subset U$ for each $i \in N$. Since $\lim_{i\to\infty} b^{(2)}_{m,p,i} = a^{(3)}_{m,p}$ and U is closed, $a^{(3)}_{m,p} \in U$. This is impossible as $I^{(3)}_{m,p} \subset U_+$. $\square$

Proposition 2.8. $\widetilde{R}_0$ *from Example* 2.6 *is a basic Urysohn bunch.*

Proof. Let $I_0, I_1, \ldots$ be any enumeration of $\mathcal{I}_0$ defined in Example 2.6. For each $n \in N$, we will define $g(n)$ in the following way. If $I_n = I^{(1)}_j$, then define $g(n)$ to be the integer such that $I_{g(n)} = I^{(1)}_{j+1}$. If $I_n = I^{(p)}_{j,k}$ (where $p = 0, 3$), then define $g(n)$ to be the integer such that $I_{g(n)} = I^{(p)}_{j,k+1}$. If $I_n = I^{(q)}_{j,k,i}$ (where $q = 2, 4$), then define $g(n)$ to be the integer such that $I_{g(n)} = I^{(q)}_{j,k,i+1}$. In other words, $I_{g(n)}$ is the first segment in $\mathcal{I}_0$ you encounter going downward from I_n (see Fig. 3). Let a_n be the left endpoint of I_n, and let b_n be the right endpoint. Set $c_n = a_n$ and $c^+_n = a_{g(n)}$ if $S_n \subset S^{(0)} \cup S^{(1)} \cup S^{(2)} \cup S^{(3)}$. Set $c_n = b_n$ and $c^+_n = b_{g(n)}$ if $S_n \subset S^{(4)}$. To verify conditions (13) and (14) of Definition 2.4, set $G = I \cup (R_0 \cap S^{(0)})$, $G_+ = I_+ \cup (R_0 \cap S^{(4)})$, $F = G \cup (R_0 \cap S^{(1)})$ and $F_+ = G_+ \cup (R_0 \cap S^{(3)})$.

Condition (8) of Definition 2.4 was proven in Proposition 2.7. We leave verifying the remainder of 2.4 to the reader. (Note that the rectangle S_n is uniquely determined by conditions (4) and (5).) $\square$

3. Construction of $\widetilde{X}$

In this section we assume that $\widetilde{X}_0$ is a basic Hausdorff (or Urysohn) bunch as described in Definition 2.4. We use the notation from 2.4 and construct a connected Hausdorff (Urysohn) countable bunch $\widetilde{X}$ of free straight linear segments contained in the plane $\mathbb{R}^2$ (see Fig. 4).

Proposition 3.1. *Let K be a subset of N and let $Y = \bigcup_{k\in K} I_k$. Suppose that, for $k \in K$, $L_k \subset S_k \cup S_{\hat{g}(k)}$. Let $Z = Y \cup \bigcup_{k\in K} L_k$. Then,* $\mathrm{Cl}(Z) = \mathrm{Cl}(Y) \cup \bigcup_{k\in K} \mathrm{Cl}(L_k)$.

Proof. Take a sequence $z_0, z_1, \ldots$ of points of Z converging to a point t. We will prove that $t \in \mathrm{Cl}(Y) \cup \bigcup_{k\in K} \mathrm{Cl}(L_k)$. Let $k(n)$ be an element of K such that $z_n \in L_{k(n)} \cup I_{k(n)}$. If some integer k repeats infinitely many times in the sequence $k(0), k(1), \ldots$, then it follows that $t \in \mathrm{Cl}(L_k) \cup I_k$. Thus, we may assume that the sequence $k(0), k(1), \ldots$ is not repetitive. By 2.4(4) and (5), there is $y_n \in I_{k(n)}$ such that the straight linear segment between y_n and z_n is vertical ($y_n = z_n$ if $z_n \in I_{k(n)}$). It follows from 2.4(3) that $\lim_{n\to\infty} y_n = t$. Therefore, $t \in \mathrm{Cl}(Y)$. $\square$

Set $T = X_0 \cup \bigcup_{n\in N} \mathrm{Int}(S_n)$ and $\widetilde{T} = T \cup I \cup I_+$. The following proposition is an easy consequence of 3.1 and 2.4(4), (5) and (7).

Proposition 3.2. $\mathrm{Cl}(T) = T \cup \{c, c^+\} \cup \bigcup_{n\in N} J_n$ *where* $J_n = \mathrm{Bd}(S_n) \setminus (I_n \cup I_{g(n)})$. *In particular,* $I \cap \mathrm{Cl}(T) = \{c\}$ *and* $I_+ \cap \mathrm{Cl}(T) = \{c^+\}$.

Proposition 3.3. *Suppose C is a subset of $\widetilde{T}$ containing $\widetilde{X}_0$. Let W be an open subset of $\widetilde{X}_0$ saturated by $\widetilde{\mathcal{I}}_0$. Let $U \subset C$ be such that $U \cap \widetilde{X}_0 = W$ and the following two conditions hold for each $n \in N$:*

(1) *$S_n \cap U$ is open in $S_n \cap C$.*
(2) *$S_n \cap C \subset U$ if both I_n and $I_{g(n)}$ are contained in W.*

Then, U is open in C.

Proof. Let $K = \{k \in N \mid I_k \cap W = \emptyset\}$. For each $k \in K$, let $L_k = (C \setminus U) \cap (S_k \cup S_{\hat{g}(k)})$. Observe that L_k is closed in C and $C \setminus U = (\widetilde{X}_0 \setminus W) \cup \bigcup_{k \in K} L_k$. Since $\widetilde{X}_0$ is closed in $\widetilde{T}$, the set $\widetilde{X}_0 \setminus W$ is closed in $\widetilde{T}$ and, consequently, it is closed in C. It follows from 3.1 that $C \setminus U$ is closed in C. □

Let W be an arbitrary subset of $\widetilde{X}_0$ saturated by $\widetilde{\mathcal{I}}_0$. Let $T(W)$ denote the union of W and the interiors of all S_n's such that $I_n \cup I_{g(n)} \subset W$. Set $G^* = T(G)$ and $G^*_+ = T(G_+)$. Using 3.3 and 2.4(13) we infer the following proposition.

Proposition 3.4. *G^* and G^*_+ are open in $\widetilde{T}$, $I \subset G^*$, $I_+ \subset G^*_+$, and $G^* \cap G^*_+ = \emptyset$.*

The following proposition is an easy corollary of 3.1.

Proposition 3.5. *Suppose that $\widetilde{X}_0$ is a basic Urysohn bunch an F and F_+ are as in* 2.4(14). *Then $F^* = G^* \cup F$ and $F^*_+ = G^*_+ \cup F_+$ are closed in $\widetilde{T}$.*

For each $n \in N$, let h_n denote the affine map of the plane onto itself such that

(h-1) $h_n(S) = S_n$,
(h-2) $h_n(c) = c_n$, and
(h-3) $h_n(c^+) = c_n^+$.

Clearly,

(h-4) $h_n(I) = I_n$ is the upper side of S_n, and
(h-5) $h_n(I_+)$ is the lower side of S_n, and it is contained in $I_{g(n)}$.

As we promised in the previous section, we will recursively paste together infinitely many mutually exclusive copies of X_0 to get a connected bunch of arcs X. The following notation will allow us to name the elements of X introduced in each step of the construction.

For each integer $i \in N$, let $\mathcal{P}_i$ be the collection of all sequences of integers $(n_0, n_1, \ldots, n_i)$ with $n_k \in N$ for $k = 0, \ldots, i$. Let $\mathcal{P} = \bigcup_{i=0}^{\infty} \mathcal{P}_i$. For $\varepsilon = (n_0, n_1, \ldots, n_i) \in \mathcal{P}_i$ we will use the following notation:

$$\varepsilon' = (n_0, n_1, \ldots, n_{i-1})$$

and

$$\varepsilon'' = n_i.$$

Additionally, if j is an integer between 0 and i, we set

$$s(j, \varepsilon) = (n_0, n_1, \ldots, n_j).$$

Also, we define $s(-1, \varepsilon)$ to be the empty set. Observe that $\varepsilon' = s(i-1, \varepsilon)$, $\varepsilon'' \in \mathcal{P}_0$, ε' is empty for $i = 0$, and $\varepsilon' \in \mathcal{P}_{i-1}$ for $i > 0$.

We will write $\eta \prec \varepsilon$ for $\eta, \varepsilon \in \mathcal{P}$, if $\eta \neq \varepsilon$ and $\eta = s(j, \varepsilon)$ for some integer j.

For convenience we will often disregard the parentheses around elements of $\mathcal{P}$. In particular, if $\varepsilon = (n_0, n_1, \ldots, n_i)$, then by (ε, n_{i+1}) we will understand the sequence $(n_0, n_1, \ldots, n_i, n_{i+1})$. If σ is another element of $\mathcal{P}$, then by $\varepsilon\sigma = (\varepsilon, \sigma)$ we denote the concatenation of ε and σ, that is the sequence formed from ε followed by σ.

We will also frequently write elements of $\mathcal{P}$ without the parentheses. For example, we will simply write a_n instead of $a_{(n)}$. According to this convention, we can interpret previously defined a_n, b_n, I_n, S_n, c_n, c_n^+ and h_n as, respectively, a_ε, b_ε, I_ε, S_ε, c_ε, c_ε^+ and h_ε, for $\varepsilon = (n) \in \mathcal{P}_0$. We will now define a_ε, b_ε, I_ε, S_ε, c_ε, c_ε^+ and h_ε for $\varepsilon \in \mathcal{P}_i$ with $i > 0$. We will start with recursively defining $h_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$. Suppose $i > 0$ and h_η have been already defined for each $\eta \in \mathcal{P}_{i-1}$. For any $\varepsilon \in \mathcal{P}_i$, let

$$h_\varepsilon = h_{\varepsilon'} \circ h_{\varepsilon''}.$$

Now, set $a_\varepsilon = h_{\varepsilon'}(a_{\varepsilon''})$, $b_\varepsilon = h_{\varepsilon'}(b_{\varepsilon''})$, $I_\varepsilon = h_{\varepsilon'}(I_{\varepsilon''})$, $S_\varepsilon = h_{\varepsilon'}(S_{\varepsilon''})$, $c_\varepsilon = h_{\varepsilon'}(c_{\varepsilon''})$ and $c_\varepsilon^+ = h_{\varepsilon'}(c_{\varepsilon''}^+)$.

Let h denote the identity on $\mathbb{R}^2$. It will be convenient to set $h_{\varepsilon'} = h$ for $\varepsilon \in \mathcal{P}_0$ (recall that for each $\varepsilon \in \mathcal{P}_0$, ε' is empty). Observe that, in this notation, the formulas defining h_ε, a_ε, b_ε, I_ε, S_ε, c_ε and c_ε^+ for $\varepsilon \in \mathcal{P}_i$ with $i > 0$ are also valid for $\varepsilon \in \mathcal{P}_0$. In particular, the equality

$$h_\varepsilon = h_{\varepsilon'} \circ h_{\varepsilon''} \tag{$*$}$$

holds for any $\varepsilon \in \mathcal{P}$.

Applying $(*)$ $i-1$ times to $\varepsilon = (n_0, n_1, \ldots, n_i)$, we get that $h_\varepsilon = h_{n_0} \circ h_{n_1} \circ \cdots \circ h_{n_i}$. It follows that

$$h_{\alpha\beta} = h_\alpha \circ h_\beta \tag{$**$}$$

for any α and β in $\mathcal{P}$.

For each $\varepsilon \in \mathcal{P}$, let $g(\varepsilon)$ denote the sequence $(\varepsilon', g(\varepsilon''))$. Clearly, $g(\varepsilon) \in \mathcal{P}_i$ if $\varepsilon \in \mathcal{P}_i$. Observe also the above definition of $g(\varepsilon)$ coincides with previously defined $g(n)$ for $\varepsilon = (n)$.

Proposition 3.6. *If $\varepsilon \in \mathcal{P}_i$, then*

(1) *h_ε is an affine map, S_ε is a rectangle with sides horizontal and vertical, I_ε is a straight linear segment between a_ε and b_ε,*
(2) *$S_\varepsilon = h_\varepsilon(S) \subset \mathrm{Int}(S)$ for $i > 0$,*
(3) *$h_\varepsilon(I) = I_\varepsilon$ is a side (not necessary horizontal) of S_ε,*

(4) $S_\varepsilon \cap S_{g(\varepsilon)} = h_\varepsilon(I_+)$ *is the side of* S_ε *opposite to* I_ε*, and it is contained in* $I_{g(\varepsilon)}$ (*which is a side of* $S_{g(\varepsilon)}$),
(5) $h_\varepsilon(c) = c_\varepsilon$,
(6) $h_\varepsilon(c^+) = c_\varepsilon^+$ *is either* $a_{g(\varepsilon)}$ *or* $b_{g(\varepsilon)}$.

Proof. Consider the set $\mathcal{R}$ of all rectangles in the plane with horizontal and vertical sides. Any affine map f of $\mathbb{R}^2$ onto itself such that $f(r_0) \in \mathcal{R}$ for some $r_0 \in \mathcal{R}$, has the property that $f(r) \in \mathcal{R}$ for each $r \in \mathcal{R}$. Since the class of affine maps preserving $\mathcal{R}$ is closed under the composition, (1) follows by induction from $(*)$.

(2): By the definition $S_\varepsilon = h_{\varepsilon'}(S_{\varepsilon''})$ and $S_{\varepsilon''} = h_{\varepsilon''}(S)$ by (h-1). Thus, $S_\varepsilon = h_{\varepsilon'}(h_{\varepsilon''}(S)) = h_\varepsilon(S)$ by $(*)$. To prove that $h_{\varepsilon'}(h_{\varepsilon''}(S)) \subset \operatorname{Int}(S)$ use induction with respect to i (observe that $h_{\varepsilon''}(S) \subset \operatorname{Int}(S)$ and $h_{\varepsilon'}(S) \subset S$ either trivially for $i = 0$, and by the induction for $i > 0$).

(3) and (4): By the definition $I_\varepsilon = h_{\varepsilon'}(I_{\varepsilon''})$ and $I_{\varepsilon''} = h_{\varepsilon''}(I)$ by (h-4). Thus, $I_\varepsilon = h_{\varepsilon'}(h_{\varepsilon''}(I)) = h_\varepsilon(I)$ by $(*)$. Since I and I_+ are opposite sides of S, $h_\varepsilon(I) = I_\varepsilon$ and $h_\varepsilon(I_+) = h_{\varepsilon'}(h_{\varepsilon''}(I_+))$ are opposite sides of $S_\varepsilon = h_\varepsilon(S)$. Since $S_{\varepsilon''} \cap S_{g(\varepsilon'')}$ is the lower side of $S_{\varepsilon''}$ (see 2.4(15)), it follows from (h-5) that

$$S_{\varepsilon''} \cap S_{g(\varepsilon'')} = h_{\varepsilon''}(I_+) \subset I_{g(\varepsilon'')}.$$

By applying $h_{\varepsilon'}$ to the above inclusion we get

$$h_{\varepsilon'}(S_{\varepsilon''} \cap S_{g(\varepsilon'')}) = h_{\varepsilon'}(S_{\varepsilon''}) \cap h_{\varepsilon'}(S_{g(\varepsilon'')}) = S_\varepsilon \cap S_{g(\varepsilon)} = h_\varepsilon(I_+) \subset I_{g(\varepsilon)}.$$

We leave the proof of (5) and (6) to the reader. □

Proposition 3.7. *Suppose* $\varepsilon \in \mathcal{P}_i$ *and* $\eta \in \mathcal{P}_j$ *with* $j \leqslant i$. *Then, the following statements are true*:

(1) $S_\varepsilon \subset \operatorname{Int}(S_\eta)$ *if* $\eta \prec \varepsilon$.
(2) $S_\varepsilon \cap S_\eta \neq \emptyset$ *if and only if either*
- $\eta = \varepsilon$, *or*
- $\eta \prec \varepsilon$, *or*
- $\eta = g(\varepsilon)$ (*and* $S_\varepsilon \cap S_\eta \subset I_\eta$), *or*
- $\varepsilon = g(\eta)$ (*and* $S_\varepsilon \cap S_\eta \subset I_\varepsilon$).

Proof. (1) If $\eta \prec \varepsilon$, $\varepsilon = \eta\sigma$ where $\sigma \in \mathcal{P}_{i-j}$ is the tail end of ε. Using $(**)$ and 3.6(2) we get $S_\varepsilon = h_\varepsilon(S) = h_\eta(h_\sigma(S)) \subset \operatorname{Int}(h_\eta(S)) = \operatorname{Int}(S_\eta)$.

(2) If $\eta = \varepsilon$ or $\eta \prec \varepsilon$, then $S_\varepsilon \subset S_\eta$ by (1), and $S_\varepsilon \cap S_\eta = S_\varepsilon \neq \emptyset$. If either $\eta = g(\varepsilon)$, or $\varepsilon = g(\eta)$, then $S_\varepsilon \cap S_\eta \neq \emptyset$ by 3.6(4).

Suppose that $S_\varepsilon \cap S_\eta \neq \emptyset$ and neither $\eta = \varepsilon$ nor $\eta \prec \varepsilon$. Let $\ell \leqslant j$ be the least integer such that $s(\ell, \varepsilon) \neq s(\ell, \eta)$. Let $\alpha = s(\ell, \varepsilon)$ and $\beta = s(\ell, \eta)$. Notice that $\alpha' = \beta'$ and $\alpha'' \neq \beta''$. By (3.7), $S_\varepsilon \subset S_\alpha$ and $S_\eta \subset S_\beta$. Thus, $S_\alpha \cap S_\beta \neq \emptyset$. Since $S_\alpha \cap S_\beta = h_{\alpha'}(S_{\alpha''}) \cap h_{\alpha'}(S_{\beta''}) = h_{\alpha'}(S_{\alpha''} \cap S_{\beta''})$, we get the result that $S_{\alpha''} \cap S_{\beta''} \neq \emptyset$. It follows from Definition 2.4(16) that either $\alpha'' = g(\beta'')$ or $\beta'' = g(\alpha'')$. By applying $h_{\alpha'}$ to the last two equalities, we infer that either $\alpha = g(\beta)$ or $\beta = g(\alpha)$. Now, it follows from (1) and 3.6(4) that $\varepsilon = \alpha$ and $\eta = \beta$ because, otherwise, $S_\varepsilon \cap S_\eta$ would be empty. □

Proposition 3.8. *Suppose $\varepsilon, \eta \in \mathcal{P}$. Then, the following are equivalent*:

(1) $\eta \prec \varepsilon$.
(2) $I_\varepsilon \subset \mathrm{Int}(S_\eta)$.
(3) $I_\varepsilon \cap \mathrm{Int}(S_\eta) \neq \emptyset$.

Proof. The implication (1) $\Rightarrow$ (2) is a simple consequence of the following two inclusions: $I_\varepsilon \subset S_\varepsilon$ (see 3.6(3) and $S_\varepsilon \subset \mathrm{Int}(S_\eta)$ by 3.7(1). The implication (1) $\Rightarrow$ (2) is obvious.

(3) $\Rightarrow$ (1): Suppose $I_\varepsilon \cap \mathrm{Int}(S_\eta) \neq \emptyset$. Then $S_\varepsilon \cap S_\eta \neq \emptyset$ and it follows from 3.7(2) that either $\eta \prec \varepsilon$ or $\varepsilon \prec \eta$ or $\eta = \varepsilon$ or $\eta = g(\varepsilon)$ or $\varepsilon = g(\eta)$. We will exclude the last four cases, leaving only $\eta \prec \varepsilon$.

Cases $\varepsilon \prec \eta$ and $\eta = \varepsilon$: In either of the cases $S_\eta \subset S_\varepsilon$ (see 3.7(1)) and, consequently, $I_\varepsilon \cap \mathrm{Int}(S_\eta) \subset I_\varepsilon \cap \mathrm{Int}(S_\varepsilon)$. Since $I_\varepsilon \cap \mathrm{Int}(S_\varepsilon) = \emptyset$ by 3.6(3), we get a contradiction with (3).

Cases $\eta = g(\varepsilon)$ and $\varepsilon = g(\eta)$: In either of the cases, it follows from 3.6(4) that $S_\varepsilon \cap S_\eta = \mathrm{Bd}(S_\varepsilon) \cap \mathrm{Bd}(S_\eta)$. Thus, $S_\varepsilon \cap \mathrm{Int}(S_\eta) = \emptyset$. Since $I_\varepsilon \subset S_\varepsilon$, $I_\varepsilon \cap \mathrm{Int}(S_\eta) = \emptyset$ contrary to (3). □

Proposition 3.9. *Suppose $\varepsilon, \eta \in \mathcal{P}$. Then, $I_\varepsilon \cap I_\eta \neq \emptyset$ if and only if $\varepsilon = \eta$.*

Proof. Suppose $\varepsilon \in \mathcal{P}_i$ and $\eta \in \mathcal{P}_j$. We may assume without loss of generality that $j \leqslant i$.

Suppose $I_\varepsilon \cap I_\eta \neq \emptyset$. Then $S_\varepsilon \cap S_\eta \neq \emptyset$ and it follows from 3.7(2) that either $\eta = \varepsilon$ or $\eta \prec \varepsilon$ or $\eta = g(\varepsilon)$ or $\varepsilon = g(\eta)$. We will exclude the last three cases, leaving only $\eta = \varepsilon$.

Case $\eta \prec \varepsilon$: In this case $I_\varepsilon \subset \mathrm{Int}(S_\eta)$ by 3.8. Since $I_\eta \cap \mathrm{Int}(S_\eta) = \emptyset$ (see 3.6(3)), $I_\varepsilon \cap I_\eta = \emptyset$, a contradiction.

Cases $\eta = g(\varepsilon)$ and $\varepsilon = g(\eta)$: In either of the cases, $I_\varepsilon \cap I_\eta = \emptyset$ by 3.6(3) and (4). □

Let $\mathcal{I}_i = \{I_\varepsilon \mid \varepsilon \in \mathcal{P}_i\}$ for each integer $i \geqslant 0$. Set

$$\mathcal{I} = \bigcup_{i=0}^{\infty} \mathcal{I}_i$$

and $\widetilde{\mathcal{I}} = \mathcal{I} \cup \{I, I_+\}$. Let $X_i = \bigcup_{A \in \mathcal{I}_i} A$ for each integer $i \geqslant 0$. (Observe that so defined X_0 coincides with previously defined X_0.) Set

$$X = \bigcup_{i=0}^{\infty} X_i$$

and

$$\widetilde{X} = X \cup I \cup I_+.$$

The following proposition is a simple consequence of 3.6(2) and (3).

Proposition 3.10. $X \subset \mathrm{Int}(S)$.

Proposition 3.11. *If $\varepsilon \in \mathcal{P}$, then*

(1) $h_\varepsilon(X) = X \cap \mathrm{Int}(S_\varepsilon)$,

(2) $\mathrm{Int}(S_\varepsilon)$ *is saturated by* $\mathcal{I}$,
(3) $\mathrm{Bd}(S_\varepsilon) \cap X \subset I_\varepsilon \cup I_{g(\varepsilon)}$.

Proof. (1): Let σ be an arbitrary element of $\mathcal{P}$. Using 3.6(3) and (∗∗), we infer that $h_\varepsilon(I_\sigma) = h_\varepsilon(h_\sigma(I)) = h_{\varepsilon\sigma}(I) = I_{\varepsilon\sigma} \subset X$. Thus, $h_\varepsilon(X) \subset X$. Since $X \subset \mathrm{Int}(S)$, $h_\varepsilon(X) \subset h_\varepsilon(\mathrm{Int}(S)) = \mathrm{Int}(h_\varepsilon(S)) = \mathrm{Int}(S_\varepsilon)$ and, consequently, $h_\varepsilon(X) \subset X \cap \mathrm{Int}(S_\varepsilon)$. To prove the opposite inclusion take $\alpha \in \mathcal{P}$ such that $I_\alpha \cap \mathrm{Int}(S_\varepsilon) \neq \emptyset$. It follows from 3.8 that $\varepsilon \prec \alpha$. Let β denote the tail end of α such that $\alpha = \varepsilon\beta$. Using again 3.6(3) and (∗∗), we infer that $h_\varepsilon(I_\beta) = h_\varepsilon(h_\beta(I)) = h_{\varepsilon\beta}(I) = h_\alpha(I) = I_\alpha$. Notice that we not only proved that $X \cap \mathrm{Int}(S_\varepsilon) \subset h_\varepsilon(X)$ concluding the proof of (3.11), but we also proved that I_α must be contained in $h_\varepsilon(X) = X \cap \mathrm{Int}(S_\varepsilon)$ if I_α intersects $\mathrm{Int}(S_\varepsilon)$. Thus we also proved (2). (3) follows from 3.7(2) and say 3.8. □

Proposition 3.12. *Suppose U is an open and closed subset of X. Let $\alpha, \beta \in \mathcal{P}$ be such that $\alpha' = \beta'$ and $I_\alpha \subset U$. Then, $I_\beta \subset U$.*

Proof. Let $V = {h_{\alpha'}}^{-1}(U) \cap X$. Since $X \subset {h_{\alpha'}}^{-1}(X)$ (see 3.11(1)), V is open and closed in X. Let $M = \{n \in N \mid I_n \subset V\}$. As $\alpha'' \in M$, $M \neq \emptyset$.

Take an arbitrary $m \in M$. It follows from (h-4), (h-5) and 2.4(8) that I_m and $I_{g(m)}$ are contained in the same quasi-component of $h_m(\widetilde{X}_0)$. Since $h_m(\widetilde{X}_0) \subset X$, I_m and $I_{g(m)}$ are contained in the same quasi-component of X. Thus, $I_{g(m)} \subset V$ and $I_{\hat{g}(m)} \subset V$ if $I_m \subset V$. Therefore, $g(M) \cup g^{-1}(M) \subset M$, and $M = N$ by 2.4(9). It follows that $I_{\beta''} \subset V$, and $I_\beta = h_{\alpha'}(I_{\beta''}) \subset h_{\alpha'}(V) = U$. □

Proposition 3.13. *X is a connected.*

Proof. Let U be a closed and open set in X such that $U \cap X_0 \neq \emptyset$. It follows from Proposition 3.12 that $X_0 \subset U$. To complete the proof, it is enough to show that $X \subset U$. Suppose, to the contrary, that $X \setminus U \neq \emptyset$. Let i be the least integer such that $X_i \setminus U \neq \emptyset$. Observe that $i > 0$. There exists $\varepsilon \in \mathcal{P}_i$ such that $I_\varepsilon \cap U = \emptyset$. Using 3.12 with $X \setminus U$ instead of U, we get that $h_{\varepsilon'}(X_0) \subset X \setminus U$. Since $I \cap \mathrm{Cl}(X_0) \neq \emptyset$ (by 2.4(8)) and $I_{\varepsilon'} = h_{\varepsilon'}$ (by 3.6(3)), we have the result that $I_{\varepsilon'} \cap \mathrm{Cl}(h_{\varepsilon'}(X_0)) \neq \emptyset$. But, since $X \setminus U$ is closed in X and contain $h_{\varepsilon'}(X_0)$, $I_{\varepsilon'}$ must intersect $X \setminus U$. Therefore, $I_{\varepsilon'}$ is not contained in U, which contradicts the choice of i. □

Proposition 3.14. *Suppose $\varepsilon_1, \varepsilon_2, \ldots,$ is a sequence of different elements of $\mathcal{P}$ and $x_j \in I_{\varepsilon_j}$ for each positive integer j. Suppose also that* $\lim_{j\to\infty} x_j = x \in I_\eta$ *for some $\eta \in \mathcal{P}$. Then, either $x = a_\eta$ or $x = b_\eta$.*

Proof. Since $I_\eta \subset \mathrm{Int}(S_{\eta'})$ (see 3.8), only finitely many elements of the sequence $x_1, x_2, \ldots$ may be outside $\mathrm{Int}(S_{\eta'})$. We may assume without loss of generality that $x_j \in \mathrm{Int}(S_{\eta'})$ for each positive integer j. Set $y = {h_{\eta'}}^{-1}(x)$ and $y_j = {h_{\eta'}}^{-1}(x_j)$. Clearly, $\lim_{j\to\infty} y_j = y$. Since $h_{\eta'}(I_{\eta''}) = I_\eta$, $y \in I_{\eta''}$. It follows from 3.11(1) that $y_j \in X$. Thus, there is $\sigma_j \in \mathcal{P}$ such that $y_j \in I_{\sigma_j}$. Since $h_{\eta'}(I_{\sigma_j}) = I_{\varepsilon_j}$, it follows from 3.6(3), 3.9 and (∗∗) that $\varepsilon_j = \eta'\sigma_j$ and, therefore the sequence $\sigma_1, \sigma_2, \ldots$ has no repetition. Since $x = h_{\eta'}(y)$, $a_\eta = h_{\eta'}(a_{\eta''})$

and $b_\eta = h_{\eta'}(b_{\eta''})$, showing the following claim will complete the proof of the proposition. $\square$

Claim 3.14.1. *y is either $a_{\eta''}$ or $b_{\eta''}$.*

Recall that $T = X_0 \cup \bigcup_{n\in N} \mathrm{Int}(S_n)$. Clearly, $X \subset T$. Let $\alpha = \hat{g}(\eta'')$ (recall that $\hat{g}(\eta'') = g^{-1}(\eta'')$ if $\eta'' \in g(N)$, and $\hat{g}(\eta'') = \eta''$ otherwise). The claim follows from 2.4(12) if $T \setminus (S_{\eta''} \cup S_\alpha)$ contains infinitely many elements of the sequence $y_1, y_2, \ldots$. So, we may assume that $y_j \in S_\beta$ where β is either η'' or α. Observe that $y \in I_\beta$ if $\beta = \eta''$ and $y \in I_{g(\beta)}$ otherwise. Since the sequence $\sigma_1, \sigma_2, \ldots$ has no repetition, $X \cap \mathrm{Int}(S_\beta)$ contains infinitely many y_j's. 3.11(1) implies $X \cap \mathrm{Int}(S_\beta) = h_\beta(X) \subset h_\beta(T)$. By 3.6(3) and (4), $h_\beta(\mathrm{Cl}(T) \cap (I \cup I_+)) = \mathrm{Cl}(h_\beta(T)) \cap (I_\beta \cup I_{g(\beta)})$. Since $\mathrm{Cl}(T) \cap (I \cup I_+) = \{c, c^+\}$ (see 3.2), y must be either $h_\beta(c)$ or $h_\beta(c^+)$. Now, the claim follows from (h-2), (h-3), and 2.4(10) and (11).

For an arbitrary segment $A \in \mathcal{I}_0$, we will now define a set $e(A) \subset X$. Let $n \in N$ be such that $A = I_n$. Set

$$e(A) = \begin{cases} (A \cup h_n(G^*) \cup h_{\hat{g}(n)}(G_+^*)) \cap X & \text{if } n \in g(N), \\ (A \cup h_n(G^*)) \cap X & \text{if } n \notin g(N). \end{cases}$$

If $\widetilde{X}_0$ is a basic Urysohn bunch (and therefore, the sets F^* and F_+^* are defined in Proposition 3.5), we will also define $\bar{e}(A)$ by setting

$$\bar{e}(A) = \begin{cases} (A \cup h_n(F^*) \cup h_{\hat{g}(n)}(F_+^*)) \cap X & \text{if } n \in g(N), \\ (A \cup h_n(F^*)) \cap X & \text{if } n \notin g(N). \end{cases}$$

Let $W \subset \widetilde{X}_0$ be an arbitrary set saturated by $\widetilde{\mathcal{I}}_0$. Recall that $T(W)$ denote the union of W and the interiors of all S_n's such that $I_n \cup I_{g(n)} \subset W$. Let $e(W)$ denote the union of $T(W) \cap \widetilde{X}$ and all the sets $e(A)$ where $A \in \mathcal{I}_0$ and $A \subset W$. If $\widetilde{X}_0$ is a basic Urysohn bunch, then we additionally define $\bar{e}(W)$ to be the union of $T(W) \cap \widetilde{X}$ and all the sets $\bar{e}(A)$ where $A \in \mathcal{I}_0$ and $A \subset W$. Notice that Proposition 3.11(3) implies that using S_n instead of $\mathrm{Int}(S_n)$ in the definition of $T(W)$ does not change $\bar{e}(A)$.

Proposition 3.15. *If W is a subset of $\widetilde{X}_0$ saturated by $\widetilde{\mathcal{I}}_0$, then $e(W)$ is saturated by $\widetilde{\mathcal{I}}$. The same is true for $\bar{e}(W)$ if it is defined.*

Proposition 3.16. *Let W_0 and W_1 be disjoint subsets of $\widetilde{X}_0$ saturated by $\widetilde{\mathcal{I}}_0$. Then, $e(W_0) \cap e(W_1) = \emptyset$. Moreover, $\bar{e}(W_0) \cap \bar{e}(W_1) = \emptyset$ (if $\widetilde{X}_0$ is a basic Urysohn bunch).*

Using 3.3 (with $C = \widetilde{X}$) and 3.4 we infer the following proposition.

Proposition 3.17. *Let W be an open subset of $\widetilde{X}_0$ saturated by $\widetilde{\mathcal{I}}_0$. Then, $e(W)$ is open in $\widetilde{X}$.*

Proposition 3.18. *Suppose $\widetilde{X}_0$ is a basic Urysohn bunch and W is a closed subset of $\widetilde{X}_0$ saturated by $\widetilde{\mathcal{I}}_0$. Then, $\bar{e}(W)$ is closed in $\widetilde{X}$.*

Proof. Notice that Proposition 3.11(3) implies that using S_n instead of $\mathrm{Int}(S_n)$ in the definition of $T(W)$ does not change $\bar{e}(A)$. Consequently, we may use S_n instead of $\mathrm{Int}(S_n)$ in the construction of G^* and G^*_+ without changing $\bar{e}(A)$. It follows from 3.2, 3.5 and 3.11(3) that $I_n \cup h_n(F^*) \cap X$ is closed in $\widetilde{X}$ for each $n \in N$. Similarly, $I_n \cup h_{\hat{g}(n)}(F^*_+) \cap X$ is closed in $\widetilde{X}$ for each $n \in g(N)$.

Let $K \subset N$ denote the set of those integers k such that $I_k \subset W$. For $k \in K$, let L'_k denote $S_k \cap X$ if $g(k) \in K$, and let $L'_k = I_k \cup h_k(F^*) \cap X$ otherwise. Let $L''_k = \emptyset$ if $k \notin g(N)$, else let L''_k denote $S_{\hat{g}(k)} \cap X$ if $\hat{g}(k) \in K$. Finally, let $L''_k = I_k \cup h_{\hat{g}(k)}(F^*_+) \cap X$ if $k \in g(N)$ and $\hat{g}(k) \notin K$. Set $L_k = L'_k \cup L''_k$. Observe that L_k is closed in $\widetilde{X}$ and $\bar{e}(W) = W \cup \bigcup_{k\in K} L_k$. Now, the proposition follows from Proposition 3.1. □

Proposition 3.19. *$\widetilde{X}$ is a Hausdorff bunch of arcs. Moreover, $\widetilde{X}$ is Urysohn if $\widetilde{X}_0$ is a basic Urysohn bunch.*

Proof. Let α and β be two different elements of $\mathcal{P}$. We will show that there are two disjoint sets G_α and G_β open in $\widetilde{X}$, saturated by $\widetilde{\mathcal{I}}$, and such that $I_\alpha \subset G_\alpha$ and $I_\beta \subset G_\beta$. Additionally, in the case $\widetilde{X}_0$ is a basic Urysohn bunch, we will show that there are two disjoint sets F_α and F_β closed in $\widetilde{X}$, saturated by $\widetilde{\mathcal{I}}$, and such that $G_\alpha \subset F_\alpha$ and $G_\beta \subset F_\beta$.

Case $\alpha' = \beta'$: By 2.4(8), there is a set W closed and open in $\widetilde{X}_0$ such that $I_{\alpha''} \in W$ and $I_{\beta''} \in \widetilde{X}_0 \setminus W$. Set

$$G_\alpha = h_{\alpha'}\bigl(e(W) \setminus (I \cup I_+)\bigr) \quad \text{and}$$
$$G_\beta = h_{\alpha'}\bigl(e(\widetilde{X}_0 \setminus W) \setminus (I \cup I_+)\bigr).$$

Since $h_{\alpha'}$ restricted to X is a homeomorphism of X onto $\mathrm{Int}(S_{\alpha'}) \cap X$ and $\mathrm{Int}(S_{\alpha'}) \cap X$ is saturated by $\mathcal{I}$ (see 3.6(1) and 3.11), we infer that

- $I_\alpha \subset G_\alpha$ and $I_\beta \subset G_\beta$ (see 3.6(3)),
- G_α and G_β are saturated by $\mathcal{I}$ (see 3.15),
- G_α and G_β are disjoint (see 3.16) and open (see 3.17).

Now, suppose $\widetilde{X}_0$ is a basic Urysohn bunch. Observe that I_+ is contained either in W or in $\widetilde{X}_0 \setminus W$. By swapping if necessary α with β we may assume that $I_+ \subset W$. Set $F_\alpha = h_{\alpha'}(\bar{e}(W)) \cup I_{g(\alpha')}$ and $F_\beta = h_{\alpha'}(\bar{e}(\widetilde{X}_0 \setminus W))$. (If $\alpha' = \emptyset$ we simply set $F_\alpha = \bar{e}(W)$ and $F_\beta = \bar{e}(\widetilde{X}_0 \setminus W)$.) Clearly, $G_\alpha \subset F_\alpha$ and $G_\beta \subset F_\beta$. Including $I_{g(\alpha')}$ in F_α and 3.15 guarantee that both F_α and F_β are saturated by $\widetilde{\mathcal{I}}$. $F_\alpha \cap F_\beta = \emptyset$ by 3.16. It follows from 3.11 and 3.18 that F_α and F_β are closed in $\widetilde{X}$.

Case $\alpha' \neq \beta'$: Let i and j be such that $\alpha \in \mathcal{P}_i$ and $\beta \in \mathcal{P}_j$. We may assume that $j \leqslant i$. Observe that $I_\beta \cap \mathrm{Int}(S_{\alpha'}) = \emptyset$. By 2.4(8), there is a set W closed and open in $\widetilde{X}_0$ such that $I_{\alpha''} \in W$ and $(I \cup I_+) \cap W = \emptyset$. Set $G_\alpha = h_{\alpha'}(e(W))$ and $G_\beta = h_{\alpha'}(e(\widetilde{X}_0 \setminus W)) \cup (\widetilde{X} \setminus S_{\alpha'})$. If $\widetilde{X}_0$ is a basic Urysohn bunch, then set $F_\alpha = h_{\alpha'}(\bar{e}(W))$ and $F_\beta = h_{\alpha'}(\bar{e}(\widetilde{X}_0 \setminus W)) \cup (\widetilde{X} \setminus S_{\alpha'})$. Observe that $F_\beta = h_{\alpha'}(\bar{e}(\widetilde{X}_0 \setminus W)) \cup (\widetilde{X} \setminus \mathrm{Int}(S_{\alpha'}))$. We leave the proof that so defined G_α, G_β, F_α and F_β have the required properties to the reader. □

Combining 3.6(1), 3.9, 3.13, 3.14 and 3.19 we get the following corollary.

Corollary 3.20. *Both X and $\widetilde{X}$ are connected countable Hausdorff bunches of free straight linear segments contained in $\mathbb{R}^2$. Both X and $\widetilde{X}$ are Urysohn if $\widetilde{X}_0$ is a basic Urysohn bunch.*

Using 2.5 and 2.3 we get the following corollary.

Corollary 3.21. *Let $\widetilde{P}$ (and P) denote $\widetilde{X}$ (and X, respectively) resulting from the construction with $\widetilde{X}_0$ equal to $\widetilde{P}_0$ from Example* 2.1. *Then, both P and $\widetilde{P}$ are connected countable Hausdorff, but not Urysohn, bunches of free straight linear segments in $\mathbb{R}^2$.*

Finally, 2.8 implies the following corollary.

Corollary 3.22. *Let $\widetilde{R}$ (and R) denote $\widetilde{X}$ (and X, respectively) resulting from the construction with $\widetilde{X}_0$ equal to $\widetilde{R}_0$ from Example* 2.6. *Then, both R and $\widetilde{R}$ are connected countable Urysohn bunches of free straight linear segments in $\mathbb{R}^2$.*

4. Quasi-graphs

Definition 4.1. Let Q be a metric space with two finite collections of subsets $\mathcal{V}$ and $\mathcal{E}$. We will say that Q is a *quasi-graph* with the set of *quasi-vertices* $\mathcal{V}$ and the set of *quasi-edges* $\mathcal{E}$ provided that the following conditions are satisfied:

(1) $Q = \bigcup_{V \in \mathcal{V}} V \cup \bigcup_{E \in \mathcal{E}} E$.
(2) Elements of $\mathcal{V}$ are mutually disjoint continua.
(3) Elements of $\mathcal{E}$ are closed in Q.
(4) The intersection of two distinct elements of $\mathcal{E}$ is either empty or it is an element of $\mathcal{V}$.
(5) If a quasi-edge $E \in \mathcal{E}$ intersect a quasi-vertex $V \in \mathcal{V}$, then $V \subset E$.
(6) Each quasi-edge $E \in \mathcal{E}$ contains exactly two distinct quasi-vertices. The two quasi-vertices are called the quasi-vertices of E. E does not intersect any other quasi-vertices.
(7) If V_1 and V_2 are two quasi-vertices of $E \in \mathcal{E}$, then V_1 and V_2 are included in the same quasi-component of E.

Definition 4.2. Suppose G is a graph with its set of vertices $\mathcal{V}_G$ and its set of edges $\mathcal{E}_G$. Suppose also that Q is a quasi-graph with its set of quasi-vertices $\mathcal{V}_Q$ and its set of quasi-edges $\mathcal{E}_Q$. We will say that Q is *modeled* on G provided that there is a 1-to-1 correspondence between $\mathcal{V}_Q$ and $\mathcal{V}_G$, and there is a 1-to-1 correspondence between $\mathcal{E}_Q$ and $\mathcal{E}_G$ such that for each quasi-edge $E_Q \in \mathcal{E}_Q$, the quasi-vertices of E_Q correspond to the vertices of the edge $E_G \in \mathcal{E}_G$ corresponding to E_Q.

Example 4.3. Suppose G is an arbitrary graph with its set of vertices $\mathcal{V}_G$ and its set of edges $\mathcal{E}_G$. Let $\widetilde{X}_0$ be a basic Hausdorff (or Urysohn) bunch (see Definition 2.4) and let $\widetilde{X}$ be the bunch resulting from the construction in the previous section. Construct a quasi-graph $Q(G, X)$ by replacing each vertex $v \in \mathcal{V}_G$ by a copy A_v of the interval $[0, 1]$. For each two vertices u and v belonging to some edge $E \in \mathcal{E}_G$, connect A_u with A_v by a copy $C(E)$ of the bunch of arcs $\widetilde{X}$. A_u and A_v are identified (in one of the two orders) with $I \subset C(E)$ and $I_+ \subset C(E)$. Set

$$Q(G, X) = \bigcup_{E \in \mathcal{E}_G} C(E).$$

Place the sets $C(E)$ so that they are closed in $Q(G, X)$ and $C(E_1) \cap C(E_2)$ is either empty or equal to A_v if v is a common vertex of E_1 and E_2 for each two different edges $E_1, E_2 \in \mathcal{E}_G$. Observe that $Q(G, X)$ is a quasi-graph with the set of vertices $\{A_v \mid v \in \mathcal{V}_G\}$ and the set of edges $\{C(E) \mid E \in \mathcal{E}_G\}$. Observe also that so constructed $Q(G, X)$ is a countable connected Hausdorff (or Urysohn) bunch of free arcs.

Let ξ_E denote the defining homeomorphism of $\widetilde{X}$ onto $C(E)$. Set $C_0(E) = \xi_E(\widetilde{X}_0)$ for each $E \in \mathcal{E}_G$. By 2.4(8),

$$Q(G, X_0) = \bigcup_{E \in \mathcal{E}_G} C_0(E)$$

is a quasi-graph with vertices $\{A_v \mid v \in \mathcal{V}_G\}$ and edges $\{C_0(E) \mid E \in \mathcal{E}_G\}$. So constructed $Q(G, X_0)$ is a countable Hausdorff (Urysohn) bunch of free arcs. It is, however, not connected.

In particular, by taking $\widetilde{P}_0$ from Example 2.1, we get quasi-graphs $Q(G, P_0)$ and $Q(G, P)$ that are Hausdorff but not Urysohn. By taking $\widetilde{R}_0$ from Example 2.6, we get two Urysohn quasi-graphs $Q(G, R_0)$ and $R(G, R)$.

Proposition 4.4. *Let Q and Q' be countable bunches of arcs which are resolutions of the same space Y. Suppose that Q is a quasi-graph modeled on a certain graph G in such a way that quasi-vertices of Q are arcs of the bunch. Then, Q' is also a quasi-graph modeled on G.*

Proof. Let $\mathcal{V}$ be the collection of quasi-vertices of Q and let $\mathcal{E}$ be the collection of quasi-edges. Observe that each $E \in \mathcal{E}$ is saturated by the arcs in the bunch Q. Let q and q' denote the natural quotient projections of Q and Q', respectively, onto Y. For each set $S \subset Q$, let S' denote the set $(q')^{-1}(q(S))$. It is easy to verify that Q' with $\{V' \mid V \in \mathcal{V}\}$ as quasi-vertices and $\{E' \mid E \in \mathcal{E}\}$ as quasi-edges is a quasi graph modeled on G. $\square$

Lemma 4.5. *Suppose M is locally connected metric space and C is an arbitrary subset of M. Suppose K and L are two disjoint non-empty connected closed subsets of C. Then K and L are contained in the same quasi-component of C if and only if for each neighborhood U of $C \setminus (K \cup L)$ in M and for each two disjoint neighborhoods U_K and U_L in M of K and L, respectively, there is a component of U intersecting both U_K and U_L.*

Proof. *Sufficiency*: Suppose that K and L are not contained in the same quasi-component of C. Then, there is a set $V \subset C$ both open and closed in C such that K intersects V and L intersects $C \setminus V$. Since K and L are connected, $K \subset V$ and $L \subset C \setminus V$. There are two sets U_K and U_L open in M such that $V \subset U_K$, $C \setminus V \subset U_L$ and $U_K \cap U_L = \emptyset$. Let

$U = U_K \cup U_L$. Clearly, no component of U intersects both U_K and U_L, since component of U is contained either in U_K or in U_L and these sets are exclusive.

Necessity: Let U, U_K and U_L be neighborhoods in M of $C \setminus (K \cup L)$, K and L, respectively. Suppose that $U_K \cap U_L = \emptyset$ and no component of U intersects both U_K and U_L. Let W be the union of components of U intersecting U_K. Observe that the set $Y = U \setminus W$ is the union of components of U not intersecting U_K. Since a component of an open set in a locally connected space is open, both W and Y are open in M. Now, the sets $Z_K = U_K \cup W$ and $Z_L = U_L \cup Y$ are open and disjoint neighborhoods of K and L, respectively. Since $C \subset Z_K \cup Z_L$, the sets $C \cap Z_K$ and $C \cap Z_L$ are both closed and open in C. Consequently, K and L are not contained in the same quasi-component of C. □

Theorem 4.6. *Suppose that $Q \subset \mathbb{R}^2$ is a quasi-graph modeled on a graph G. Then G can be embedded in $\mathbb{R}^2$.*

Proof. We will denote by $\mathcal{V}_Q$ the collection of quasi-vertices of Q and by $\mathcal{E}_Q$ the collection of quasi-edges of Q. Similarly, let $\mathcal{V}_G$ be the set of vertices of G and let $\mathcal{E}_G$ be the set of edges of G. It will be convenient for us to assume that Q is a subset of the two-dimensional sphere S^2, and embed G also in S^2. Clearly, such a restatement of 4.6 is equivalent to 4.6.

Suppose that the theorem is not true and that G is a graph with the least possible number of vertices for which the conclusion of the theorem does not hold.

Claim 4.6.1. *No quasi-vertex of Q separates S^2 between any two other quasi-vertices of Q.*

Proof. Suppose that there exists a quasi-vertex $V \in \mathcal{V}_Q$ separating S^2 between two other quasi-vertices $V', V'' \in \mathcal{V}_Q$. Denote by D the component of $S^2 \setminus V$ containing V'. Let $\mathcal{W}'$ be the collection of all elements of $\mathcal{V}_Q$ that are contained in D. Set $\mathcal{V}'_Q = \mathcal{W}' \cup \{V\}$, $\mathcal{W}'' = \mathcal{V}_Q \setminus \mathcal{V}'_Q$ and $\mathcal{V}''_Q = \mathcal{W}'' \cup \{V\}$.

Let $\mathcal{V}'_G$ and $\mathcal{V}''_G$ be the sets of vertices of G corresponding to $\mathcal{V}'_Q$, $\mathcal{V}''_Q$, respectively. The intersection of $\mathcal{V}'_G$ and $\mathcal{V}''_G$ consists of one vertex, say V_G, corresponding to V. Observe also that $\mathcal{V}'_G \cup \mathcal{V}''_G = \mathcal{V}_G$ and each of the sets $\mathcal{V}'_G$ and $\mathcal{V}''_G$ is a proper subset of $\mathcal{V}_G$. Indeed $\mathcal{V}'_G$ does not contain the vertex corresponding to V'' and $\mathcal{V}''_G$ does not contain the vertex corresponding to V'.

We will now show that there is no quasi-edge in $\mathcal{E}_Q$ containing both an element of $\mathcal{W}'$ and an element of $\mathcal{W}''$. Suppose to the contrary that a quasi-edge E contains $V_1 \in \mathcal{W}'$ and $V_2 \in \mathcal{W}''$. It follows from 4.1(6) that $V \cap E = \emptyset$. Consequently, the set $D \cap E$ is closed and open in E, $V_1 \subset D \cap E$ and $V_2 \cap (D \cap E) = \emptyset$ which contradicts 4.1(7).

Let $\mathcal{E}'_Q$ denote the set of those quasi-edges in $\mathcal{E}_Q$ whose both quasi-vertices are in $\mathcal{V}'_Q$. Let Q' be the quasi-graph with the set of quasi-vertices $\mathcal{V}'_Q$ and its set of quasi-edges $\mathcal{E}'_Q$. Similarly, let $\mathcal{E}''_Q$ denote the set of those quasi-edges in $\mathcal{E}_Q$ whose both quasi-vertices are in $\mathcal{V}''_Q$. Let Q'' be the quasi-graph with the set of quasi-vertices $\mathcal{V}''_Q$ and its set of quasi-edges $\mathcal{E}''_Q$. Observe that $\mathcal{E}'_Q \cap \mathcal{E}''_Q = \emptyset$ and $\mathcal{E}_Q = \mathcal{E}'_Q \cup \mathcal{E}''_Q$.

Let $\mathcal{E}'_G$ and $\mathcal{E}''_G$ be the sets edges of G that correspond to the sets of quasi-edges $\mathcal{E}'_Q$ and $\mathcal{E}''_Q$, respectively. Clearly, $\mathcal{E}'_G \cap \mathcal{E}''_G = \emptyset$ and $\mathcal{E}_G = \mathcal{E}'_G \cup \mathcal{E}''_G$.

Let G' be the subgraph of G with vertices $\mathcal{V}'_G$ and edges $\mathcal{E}'_G$. Finally, let G'' be the subgraph of G with vertices $\mathcal{V}''_G$ and edges $\mathcal{E}''_G$. Clearly, Q' and Q'' are modeled on G' and G'', respectively.

By the choice of G as a graph with the least possible number of vertices for which the conclusion of the theorem does not hold, we get that both G' and G'' can be embedded in S^2.

Since the intersection of G' and G'' consists of one point (namely V_G), and $G' \cup G'' = G$, the graph G can be embedded in S^2. This contradiction concludes the proof of the claim. □

For each quasi-vertex $V \in \mathcal{V}_Q$, let $\widehat{V}$ denote the union of V and all components of $S^2 \setminus V$ not containing any elements of $\mathcal{V}_Q$. By replacing each quasi-vertex V by $\widehat{V}$, and also replacing each quasi-edge E with its vertices V' and V'' by $(E \cup \widehat{V'} \cup \widehat{V''}) \setminus \bigcup_{V \in \mathcal{V} \setminus \{V', V''\}} \widehat{V}$, we may assume that none of the quasi-vertices of Q separates S^2. Using 4.1(3)–(5), we can find a collection $\{D_V \mid V \in \mathcal{V}_Q\}$ of mutually exclusive closed disks contained in S^2 such that for each $V \in \mathcal{V}_Q$ we have

- $V \subset \mathrm{Int}(D_V)$, and
- $D_V \cap E = \emptyset$ for each $E \in \mathcal{E}_Q$ not containing V.

Set $T = \bigcup_{V \in \mathcal{V}_Q} D_V$. For each quasi-edge $E \in \mathcal{E}_Q$, let $V'(E)$ and $V''(E)$ denote the two quasi vertices contained in E (see 4.1(6)). Set $E^* = E \setminus (V'(E) \cup V''(E))$ and $T_E = T \setminus (D_{V'(E)} \cup D_{V''(E)})$.

For a real number $r > 0$ and a point $x \in S^2$, let $B(x, r)$ denote the open ball in S^2 centered at x and with radius r. It follows from 4.1(3) and (4) that for each $E \in \mathcal{E}_Q$ and each point $x \in E^*$, there is a positive number $r(x)$ such that $B(x, r(x)) \cap (T_E \cup \bigcup_{E' \in \mathcal{E}_Q \setminus \{E\}} E') = \emptyset$. Let $U_E = \bigcup_{x \in E^*} B(x, r(x)/2)$. Observe that the collection $\{U_E \mid E \in \mathcal{E}_Q\}$ consists of mutually exclusive sets each of which is open in S^2.

By Lemma 4.5, there is a component Z_E of U_E intersecting both $D_{V'(E)}$ and $D_{V''(E)}$. Since Z_E is a component of an open set in S^2, it is arcwise connected. Therefore, there is an arc $C_E \subset Z_E$ intersecting both $D_{V'(E)}$ and $D_{V''(E)}$. Let $A_E \subset C_E$ be a minimal arc intersecting both $D_{V'(E)}$ and $D_{V''(E)}$. Observe that one of the endpoints of A_E is contained in $D_{V'(E)}$ and the other in $D_{V''(E)}$. Also, $A_E \cap T_E = \emptyset$. Since $A_E \subset U_E$ and the sets $\{U_E \mid E \in \mathcal{E}_Q\}$ are mutually exclusive. Set $H = T \cup \bigcup_{E \in \mathcal{E}_Q} A_E$. Let G_Q denote the graph resulting from H by identifying each of the disks $\{D_V \mid V \in \mathcal{V}_Q\}$ to a point. Clearly, G_Q is homeomorphic to G. Now, 4.6 follows from the Moore plane decomposition theorem, since the space, resulting from S^2 by identifying to a point each of the finitely many mutually exclusive closed disks $\{D_V \mid V \in \mathcal{V}_Q\}$, is homeomorphic to S^2. □

In view of Proposition 4.4, the following corollary answers negatively the question in [5, Problem 3(b)].

Corollary 4.7. *The bunches of arcs $Q(G, P)$, $Q(G, P_0)$, $Q(G, R)$ and $Q(G, R_0)$ from Example* 4.3 *cannot be embedded in the plane if G is a graph that cannot be embedded in* *the plane. In particular,* $Q(K, P)$, $Q(K, P_0)$, $Q(K, R)$ *and* $Q(K, R_0)$ *cannot be embedded in the plane if* K *is one of the Kuratowski skew curves* (*see* [7, p. 159]).

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